ON THE BURAU REPRESENTATION FOR $n = 4$

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Abstract

The problem of faithfulness of the (reduced) Burau representation for $n = 4$ is known to be equivalent to the problem of whether certain two matrices A and B generate a free group of rank two [Bir]. In [Ber-Tra1] we gave a simple proof that $\langle A^3, B^3 \rangle$ is a free group of rank two, the result known earlier from [Wit-Zar]. In this paper we use a combination of methods of linear algebra and homology theory (the forks and noodles approach) [Ber-Tra2], [Big] to give another proof that $\langle A^3, B^3 \rangle$ is a free group and some explanations which show why we believe that $\langle A^2, B^2 \rangle$ should be a free group as well.
1 2 3 4 5

[1] [Ber-Tra1] Beridze, A.; Traczyk, P. Burau representation for $n = 4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*

[2] [Ber-Tra2] Beridze, A.; Traczyk, P. Forks, noodles and the Burau representation for $n = 4$. *Trans. A. Razmadze Math. Inst. 172 (2018), no. 3, part A, 337–353.*

[3] [Bir] Joan S Birman. Braids, links, and mapping class groups. *Annals of Mathematics Stidies, No. 82, Princeton University Press, Princeton, NJ (1974)*

[4] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

[5] [Wit-Zar] S. Witzel and M. C. B. Zaremsky. A free subgroup in the image of the

The Burau Representation, Noodles and Forks

The reduced Burau representation for $n = 4$ is the homomorphism

$$\rho : B_4 \to Aut\left(H_1\left(\tilde{D}_4; Z\right)\right) \tag{1}$$

which is defined by

$$\rho(\sigma) = \widetilde{\varphi}_*, \quad \forall \sigma \in B_4, \tag{2}$$

where $\varphi : D_4 \to D_4$ is a transformation which is representative of the element $\sigma \in B_4$. The group $H_1(\tilde{D}_4; Z)$ is a free $Z\left[t, t^{-1}\right]$–module of rank 3 [Lon-Pat], [Big], [Ber-Tra2].

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[6] [Lon-Pat] D. D.Long and M. Paton. The Burau representation is not faithful for $n \geq 6$. *Topology* **32** (1993), no. 2, 439—447

[7] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

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The basis of the $\mathbb{Z}[t, t^{-1}]$-module $H_1\left(\tilde{D}_4; \mathbb{Z}\right)$

Definition

A fork is an embedded oriented tree F in the disc D with four vertices p_0, p_i, p_j and z, where $i \neq j, i, j \in \{1, 2, 3, 4\}$ such that (see [3]):

1. F meets the puncture points only at p_i and p_j;
2. F meets the boundary ∂D_4 only at p_0;
3. All three edges of F have z as a common vertex.

[9]

[9] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

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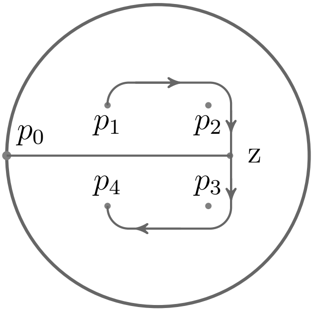
The edge of F which contains p_0 is called the **handle.** The union of the other two edges is denoted by $T(F)$ and it is called **tine of F.** Orient $T(F)$ so that the handle of F lies to the right of $T(F)$ [Big].



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[10] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

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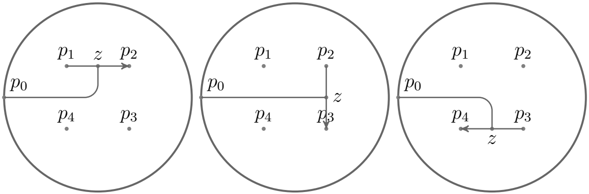
A Standard Fork

A standard fork F_i, $i = 1, 2, 3$ is the fork whose tine edge is the straight arc connecting the i-th and the (i+1)-st punctured points and whose handle has the form as in Figure below.



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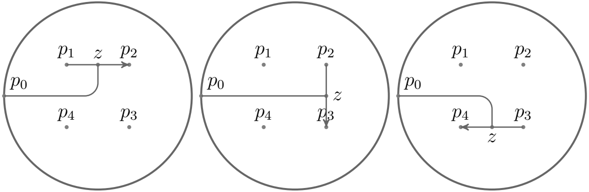

It is known that if F_1, F_2 and F_3 are the corresponding homology classes, then they form a basis of $H_1\left(\tilde{D}_4;\mathbb{Z}\right)$ [Big].
11

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The basis of the $\mathbb{Z}[t,t^{-1}]$-module $H_1\left(\tilde{D}_4;\mathbb{Z}\right)$

Using the basis derived from F_1, F_2, F_3, any automorphism

$$\widetilde{\varphi}_* : H_1\left(\tilde{D}_4;\mathbb{Z}\right) \to H_1\left(\tilde{D}_4;\mathbb{Z}\right)$$

can be viewed as a 3×3 matrix with entries in the ring $\mathbb{Z}\left[t,t^{-1}\right]$ [Big]. If $\varphi : D_4 \to D_4$ is representing an element $\sigma \in B_4$, then we need to write the matrix $\rho(\sigma) = \widetilde{\varphi}_*$ in terms of homology (algebraic) intersection pairing

$$\langle -,- \rangle : H_1\left(\tilde{D}_4, \partial\tilde{D}_4;\mathbb{Z}\right) \times H_1\left(\tilde{D}_4;\mathbb{Z}\right) \to \mathbb{Z}\left[t,t^{-1}\right].$$

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For this aim we need to define the noodles which represent relative homology classes in $H_1\left(\tilde{D}_4, \partial\tilde{D}_4;\mathbb{Z}\right)$.

12

[12] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

Definition

A noodle is an embedded oriented arc in D_4, which begins at the base point p_0 and ends at some point of the boundary ∂D_4 [Big].

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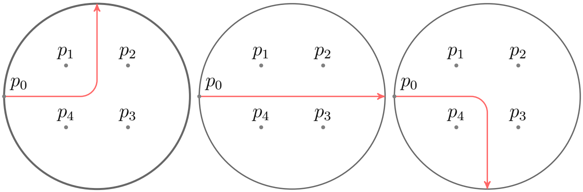

Standard noodles: N_1, N_2, N_3

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[13] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

For each $b \in H_1\left(\tilde{D}_4, \partial\tilde{D}_4; \mathbb{Z}\right)$ and $a \in H_1\left(\tilde{D}_4; \mathbb{Z}\right)$ we should take the corresponding fork N and noodle F and define the polynomial $\langle N, F\rangle \in \mathbb{Z}\left[t, t^{-1}\right]$. It does not depend on the choice of representatives of homology classes and so

$$\langle -, -\rangle : H_1\left(\tilde{D}_4, \partial\tilde{D}_4; \mathbb{Z}\right) \times H_1\left(\tilde{D}_4; \mathbb{Z}\right) \to \mathbb{Z}\left[t, t^{-1}\right]$$

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The Noodle-Fors Pairing

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The map defined by the above formula is called **the noodle–fork pairing.** [14]

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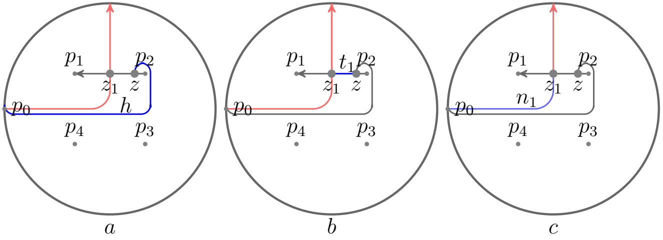
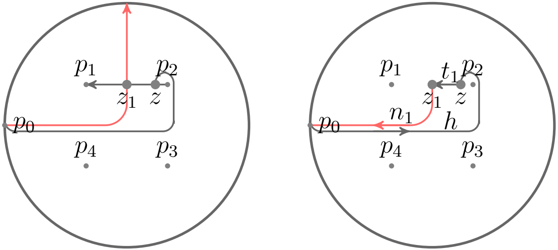
Geometric computation of noodle-forks pairing

Let $z_1, z_2, \ldots, z_n$ be the transversal intersection of N and F, ε_i be the sign of the intersection between $T(F)$ and N at z_i and $e_i = [\gamma_i]$ be the winding number of the loop γ_i around the puncture points p_1, p_2, p_3, p_4, where γ_i is the composition of three paths h, t_i and n_i: 1) h is a path from p_0 to z along the handle of F (see Figure a); 2) t_i is a path from z to z_i along the tine $T(F)$ (see Figure b); 3) n_i is a path from z_i to p_0 along the noodle N (see Figure c):



Images of the Generators σ_1, σ_2, σ_3 of the 4-Braid group B_4



$a : e_1 = -1$ $\qquad$ $b : \gamma_1 = n_1 * t_1 * h; \varepsilon_1 = 1$

The intersection of the tine $T\,(F_1\sigma_1)$ of the fork $F_1\sigma_1$ and the noodle N_1 at point z_1 is negative which means that $\varepsilon_i = -1$ (see Figure a).

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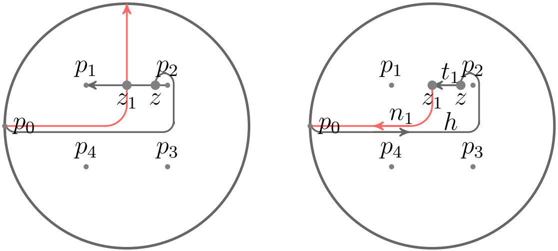

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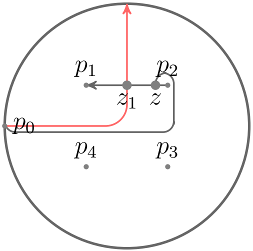

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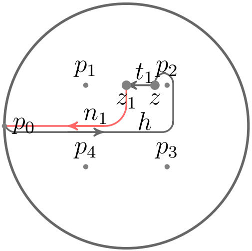

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$$\rho_{11}(\sigma_1) = \langle N_1, F_1\sigma_1 \rangle = -t$$

Geometric colsulation of $\rho_{ij}(\sigma)$

Lemma

Let $\sigma \in B_n$. Then for $1 \leq i, j \leq n-1$, the entry $\rho_{ij}(\sigma)$ of its Burau matrix $\rho(\sigma)$ is given by

$$\rho_{ij}(\sigma) = \langle N_i, F_j\sigma \rangle .$$

15 16

[15] [Big] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol. 3 (1999), 397-404*

[16] [Mat-Ito] Matthieu Calvez and Tetsuya Ito. Garside-theoretic analysis of Burau representations. *arXiv:1401.2677v2*

Under the convention adopted we have:

$$\rho\left(\sigma_1\right)=\begin{pmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho\left(\sigma_2\right)=\begin{pmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho\left(\sigma_3\right)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix}.$$

17 18

[17] [Ber-Tra.2] Beridze, A.; Traczyk, P. Forks, noodles and the Burau representation for $n=4$. *Trans. A. Razmadze Math. Inst. 172 (2018), no. 3, part A, 337–353.*

[18] [Dav] L. Davitaddze, Braid groups and Burau representation. *Bachelor thesis, Batumi, 2019*

The Bokut-Vesnin generators and kernel elements of the Burau representation

Let $\varphi : B_4 \to B_3$ be the homomorphism defined by

$$\varphi(\sigma_1) = \sigma_1, \quad \varphi(\sigma_2) = \sigma_2, \quad \varphi(\sigma_3) = \sigma_1.$$

The kernel of φ is known to be a free group $F(a, b)$ of two generators [Bok-Ves];

$$a = \sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}, \quad b = \sigma_3\sigma_1^{-1}.$$

[19] [Bok-Ves] Bokut, Leonid; Vesnin, Andrei. Gröbner-Shirshov bases for some braid groups. *J. Symbolic Comput. 41 (2006), no. 3-4, 357–371.*

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This was proved by L. Bokut and A. Vesnin [Bok-Vis].

19

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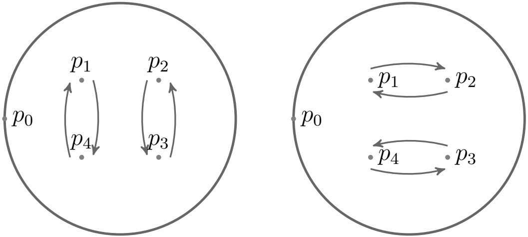
The generators a and b are in fact much more similar than they look at the first glance:



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The Bokut-Vesnin generators and kernel elements of the Burau representation

Proposition

The kernel $\ker \rho_4$ of the Burau representation map

$$\rho_4 : B_4 \to GL\left(3,\ \mathbb{Z}\left[t, t^{-1}\right]\right)$$

is a subgroup of the kernel $\ker \varphi$ of the map $\varphi : B_4 \to B_3$, defined above by

$$\varphi\left(\sigma_1\right) = \sigma_1, \quad \varphi\left(\sigma_2\right) = \sigma_2, \quad \varphi\left(\sigma_3\right) = \sigma_1.$$

Therefore

$$\ker \rho_4 \subset \ker \varphi$$

Proof. Let us make a slight detour into the realm of the Temperley-Lieb algebras TL_3 and TL_4. The Temperley-Lieb algebra TL_n is defined as an algebra over $\mathbb{Z}[t, t^{-1}]$. It has $n-1$ generators $\{U_i^i\}_{i=1}^{n-1}$, and the following relations:
(TL1) $U_i^i U_i^i = (-t^{-2} - t^2)U_i^i$,
(TL2) $U_i^i U_j^j U_i^i = U_i^i$, for $|i-j| = 1$,
(TL3) $U_i^i U_j^j = U_j^j U_i^i$, for $|i-j| > 1$.

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The Bokut-Vesnin generators and kernel elements of the Burau representation

Let us consider the homomorphism $\psi : TL_4 \to TL_3$ defined by

$$U_1^1 \to U_1^1,\ U_2^2 \to U_2^2,\ U_3^3 \to U_1^1.$$

Let $\theta : B_n \to TL_n$ be the Jones' representation defined by sending σ_i to $A + A^{-1}U_i^i$. It is known (see [Big.2], Proposition 1.5) that for $n = 3, 4$ we have $\ker \theta_n = \ker \rho_n$. Moreover, the following diagram is obviously commutative:

$$\begin{array}{ccc} B_4 & \xrightarrow{\theta_4} & TL_4 \\ \varphi \downarrow & & \downarrow \psi \\ B_3 & \xrightarrow[\theta_3]{} & TL_3 \end{array}$$

[21] [Big.2] Stephen Bigelow. Does the Jones polynomial detect the unknot? *J. Knot Theory and Ramifications (4) 11 (2002), 493-505*

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The representation θ_3 is faithful and therefore $\ker \rho_4 = \ker \theta_4 \subset \ker \varphi$. [21]

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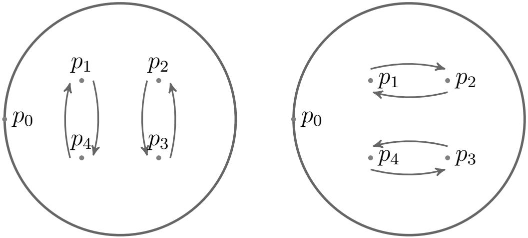
Images of The Bokut-Vesnin generators

Under the convention adopted we have:

$$A = \rho(a) = \begin{pmatrix} -t+1 & 0 & -1 \\ -t+t & -t & 0 \\ -t & 0 & 0 \end{pmatrix}, \quad B = \rho(b) = \begin{pmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix}$$



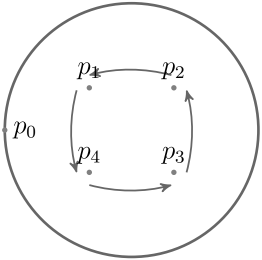
22 23

[22] [Ber-Tra.3] Beridze, A.; Traczyk, P. On the Burau representation for $n = 4$. *arXiv:1904.11730*

[23] [Dav] L. Davitaddze, Braid groups and Burau representation. *Bachelor thesis, Batumi, 2019*

Turning the punctured disk D_4 by ninety degrees

Let t be a transformation of D_4, which has the following form:



24 25

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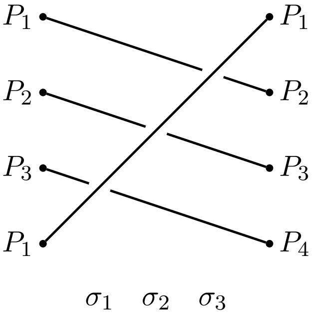
Turning the punctured disk D_4 by ninety degrees

Therefore, t is reresantatiove of the braid is $\sigma_1\sigma_2\sigma_3$:



26 27

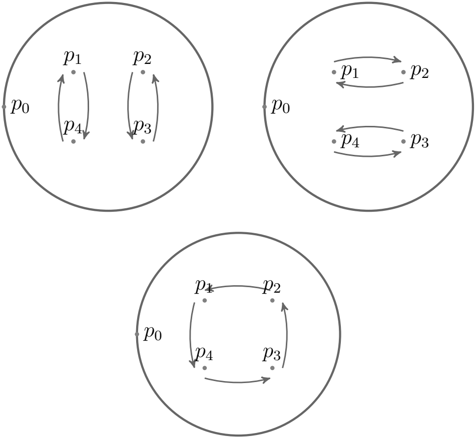
[26] [Ber-Tra.3] Beridze, A.; Traczyk, P. On the Burau representation for $n = 4$. *arXiv:1904.11730*

[27] [Dav] L. Davitaddze, Braid groups and Burau representation. *Bachelor thesis, Batumi, 2019*

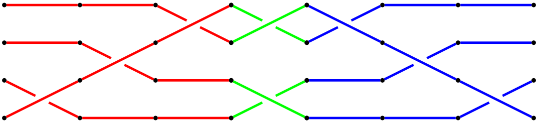
Conjugating a, a^{-1} and b^{-1} to b

It is easy to check that the following conditions are satisfied:

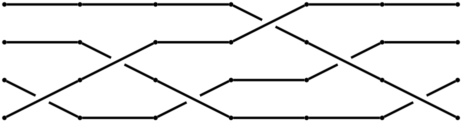
$$a = t^{-1}bt, \quad a^{-1} = tbt^{-1}, \quad b^{-1} = t^2bt^2.$$



Conjugating a^{-1} to b



$$tbt^{-1} = \sigma_1\sigma_2\sigma_3\sigma_1^{-1}\sigma_3\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$$



$$a^{-1} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}$$

Lemma

There exists a matrix T which satisfies the following equality:

$$A = TBT^{-1}, \quad A^{-1} = T^{-1}BT, \quad B^{-1} = T^2BT^2. \tag{3}$$

The considered matrix T is of order four as an element of the group $GL\left(3, \mathbb{Z}\left[t, t^{-1}\right]\right)$.

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[28] [Ber-Tra.3] Beridze, A.; Traczyk, P. On the Burau representation for $n = 4$. *arXiv:1904.11730*

Conjugating A, A^{-1} and B^{-1} to B

Proof the matrices A, B, A^{-1} and B^{-1} have the same eigenvalues $-t^{-1}, -t$ and 1. Therefore, they are conjugate to the same diagonal matrix:

$$\Delta = \begin{bmatrix} -t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t \end{bmatrix}.$$

Conjugating A, A^{-1} and B^{-1} to B

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$$\Delta = \begin{bmatrix} -t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t \end{bmatrix}.$$

Let us consider transformation matrices T_A and T_B

$$T_A = \begin{bmatrix} -1 & t^{-1} & 0 \\ -1 & t^{-1}-1 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad T_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & t^{-1}+1 & 0 \\ 0 & t^{-1} & 1 \end{bmatrix}.$$

Therefore, the following equalities hold:

$$A = T_A \Delta T_A^{-1}, \quad B = T_B \Delta T_B^{-1}.$$

Hence, we obtain that

$$A = T_A T_B^{-1} B T_B T_A^{-1}.$$

conjugating a, a^{-1} and b^{-1} to b

Let $T = T_A T_B^{-1}$, then we have:

$$T = \rho(t) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

conjugating a, a^{-1} and b^{-1} to b

Let $T = T_A T_B^{-1}$, then we have:

$$T = \rho(t) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

On the other hand, direct calculation shows that

$$T^4 = \mathbf{1}, \quad A = T^{-1}BT, \quad A^{-1} = TBT^{-1}, \quad B^{-1} = T^2BT^2.$$

Corollary

Let w be a formally irreducible non–empty word in letters A, B, A^{-1} and B^{-1} with suffix $B^i, i \geq 1$. Then the corresponding product of matrices A, B, A^{-1} and B^{-1} may be written in the form $T^{m_{k+1}} B^{n_{k+1}} T^{m_k} B^{n_k} \cdots T^{m_2} B^{n_2} T^{m_1} B^{n_1}$ $n_i \in N,\ m_i \in \{-1, 1\}$, $m_{k+1} \in \{-1, 0, 1, 2\}$.

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[30] [Ber-Tra.3] Beridze, A.; Traczyk, P. On the Burau representation for $n = 4$. *arXiv:1904.11730*

s**–pattern:** This means a one column matrix in which the position of entries with minimum degree (in the first column of the considered matrix) are checked.

[31] [Ber-Tra.1] Beridze, A.; Traczyk, P. Burau representation for $n = 4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*

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[31] [Ber-Tra.1] Beridze, A.; Traczyk, P. Burau representation for $n=4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*

s–patterns

s**–pattern:** This means a one column matrix in which the position of entries with minimum degree (in the first column of the considered matrix) are checked. For example $\begin{bmatrix} \surd \\ \circ \\ \circ \end{bmatrix}$ means that in the first column of the considered matrix there is a single entry of the smallest degree and that it is in position $(1,1)$. Altogether seven possible s–patterns exist as given below:

$$\begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix}, \begin{bmatrix} \circ \\ \surd \\ \circ \end{bmatrix}, \begin{bmatrix} \circ \\ \surd \\ \surd \end{bmatrix}, \begin{bmatrix} \surd \\ \circ \\ \circ \end{bmatrix}, \begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix}, \begin{bmatrix} \surd \\ \surd \\ \surd \end{bmatrix}$$

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[31] [Ber-Tra.1] Beridze, A.; Traczyk, P. Burau representation for $n = 4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*

Multiplication by B from the left side may affect s–patterns as illustrated in Figure:

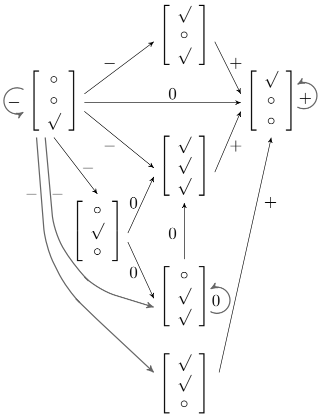

Figure 3: transformations of s–patterns — multiplication by B

transformations of s–patterns — multiplication by T

Multiplication by T may affect s–patterns as illustrated in Figure:

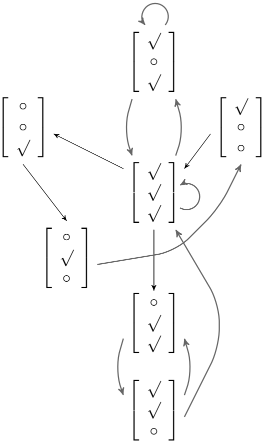

Figure 4: transformations of s–patterns — multiplication by T

Some imposible composition

Lemma

Let M be a matrix of s–pattern $\begin{bmatrix} \circ \\ \surd \\ \surd \end{bmatrix}$ or $\begin{bmatrix} \circ \\ \surd \\ \circ \end{bmatrix}$. Then the transformations corresponding to multiplication (step by step) on the left side by $BTBB$ can not be of the following form:

$$\begin{bmatrix} \circ \\ \surd \\ \surd \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \circ \\ \surd \\ \surd \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \surd \\ \surd \\ \surd \end{bmatrix} \xrightarrow{T} \begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix},$$

$$\begin{bmatrix} \circ \\ \surd \\ \circ \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \circ \\ \surd \\ \surd \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \surd \\ \surd \\ \surd \end{bmatrix} \xrightarrow{T} \begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix} \xrightarrow{B} \begin{bmatrix} \circ \\ \circ \\ \surd \end{bmatrix}.$$

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[32] [Ber-Tra.1] Beridze, A.; Traczyk, P. Burau representation for $n = 4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*

No negative loop

Main Lemma

Let ω be a word of the letters B, T and T^{-1} of the following form

$$\omega = T^{m_k} B^{n_k} \cdots T^{m_2} B^{n_2} T^{m_1} B^{n_1}, \quad n_i \in \mathbb{N},\ m_i \in \{-1, 1\}. \qquad (4)$$

If for every i we have $n_i \geq 3$, then none of transformations corresponding to any single letter in the considered word is a negative loop.

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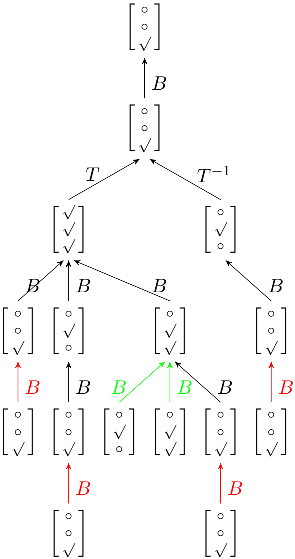
[33] [Ber-Tra.1] Beridze, A.; Traczyk, P. Burau representation for $n = 4$. *J. Knot Theory Ramifications 27 (2018), no. 3, 1840002, 6 pp.*



$$\omega = T^{n_{k+1}} B^{m_{k+1}-1} BT^{m_k} B^3 B^{n_k-3} \dots T^{m_2} B^{n_2} T^{m_1} B^{n_1}$$

Group generated by A^3 and B^3

Corollary

Let ω be a word of the form

$$\omega = T^{m_{k+1}} B^{n_k} T^{m_k} \cdots T^{m_2} B^{n_2} T^{m_1} B^{n_1} \tag{5}$$

where $n_i \geq 3$ $m_i \in \{-1, 1\}$, $m_{k+1} \in -1, 0, 1, 2$. Then the change in the minimum degree of first column entries caused by multiplication by $T^i B^r$ may be:

1. a decrease (possibly by more than 1);

2. no change at all;

3. an increase by exactly 1.

Group generated by A^3 and B^3

Corollary

Let ω be a word of the form

$$\omega = T^{m_{k+1}} B^{n_k} T^{m_k} \cdots T^{m_2} B^{n_2} T^{m_1} B^{n_1} \tag{5}$$

where $n_i \geq 3$ $m_i \in \{-1, 1\}$, $m_{k+1} \in -1, 0, 1, 2$. Then the change in the minimum degree of first column entries caused by multiplication by $T^i B^r$ may be:

1. a decrease (possibly by more than 1);

2. no change at all;

3. an increase by exactly 1.

Theorem

The matrices A^3 and B^3 generate a non–abelian free group of rank 2.

Thank You!