Burau representation for n = 4

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May 22, 2017

Abstract

The problem of faithfulness of the (reduced) Burau representation for $n = 4$ is known to be equivalent to the problem of whether certain two matrices A and B generate a free group of rank two. In this note we give a simple proof that $\langle A^3, B^3 \rangle$ is a free group of rank two.

1 Introduction.

The problem of faithfulness of the Burau representation for $n = 4$ remains the only open case of the problem in general. The representation is not faithful for $n \geq 5$ (see [2], [3], [4]) and it is faithful for $n = 1, 2, 3$. Cases $n = 1, 2$ are obvious and $n = 3$ is not very difficult.

Let us consider the following matrices:

$$b = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix}, B = b^{-1} = \begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix},$$

$$a = \begin{bmatrix} 1-t & 0 & -1 \\ t^{-1}-t & -t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix}, A = a^{-1} = \begin{bmatrix} 0 & 0 & -t^{-1} \\ 0 & -t & -t^{-1}+t \\ -1 & 0 & -t^{-1}+1 \end{bmatrix}.$$

It is known that the problem of faithfulness of the Burau representation for the braid group B_4 is equivalent to the problem of whether A and B generate a free group of rank 2 (see [1], Theorem 3.19). In this paper we give a simple proof that A^3 and B^3 generate the free group of rank two, the result known earlier from [5]. We also show that the products of the considered matrices have some special properties and we prove a minor generalisation of the main result. We hope that similar considerations used in a more refined way may prove that A^2 and B^2 generate the free group of rank two.

2 A^3, B^3 generate non–abelian free group of rank two.

We introduced the convention $A = a^{-1}, B = b^{-1}$. We recall this to explain what we mean by *non–reducible* in the Theorem below.

Theorem 2.1. *Let w be a non-reducible word in letters a, A, b, B such that wherever one of the letters appears, it is repeated at least three times consecutively. Then the corresponding product of matrices is not equal to the unit matrix. In particular A^3 and B^3 generate the free group of rank 2.*

Proof idea. We may as well assume that the word w is of the form $v \cdot bbb$, otherwise we can adjust it by conjugation. The idea is to show that the first column of the product will always contain terms t^d with negative exponents — which clearly implies that the whole matrix is not the unit matrix.

We will use schematical information of the positions occupied by terms of smallest degree (in the first column) like this: $\begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix}$ means that in the first column of the considered matrix there is a single term of smallest degree and that it is in position $(1, 1)$. We will refer to thus depicted positioning of lowest degree terms as *the s–pattern* of the matrix. Altogether one can imagine the following s–patterns

$$\begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix}, \begin{bmatrix}\circ\\ \checkmark\\ \circ\end{bmatrix}, \begin{bmatrix}\circ\\ \circ\\ \checkmark\end{bmatrix}, \begin{bmatrix}\checkmark\\ \checkmark\\ \circ\end{bmatrix}, \begin{bmatrix}\checkmark\\ \circ\\ \checkmark\end{bmatrix}, \begin{bmatrix}\circ\\ \checkmark\\ \checkmark\end{bmatrix}, \begin{bmatrix}\checkmark\\ \checkmark\\ \checkmark\end{bmatrix}.$$

They are all possible in the general case but only some of them will appear in the context of the Theorem.

We will need to study possible changes to the s–pattern when the product of matrices is multiplied from the left by one more matrix. It is clear that the smallest degree may only go down by one, stay unchanged or go up by one. There are some cases when the lowest degree certainly goes down *and* the s–pattern is either preserved or changed in a controlled way. We record this cases schematically below.

$$B\begin{bmatrix}\circ\\ \circ\\ \checkmark\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\circ\\ \circ\\ \checkmark\end{bmatrix}, B\begin{bmatrix}\circ\\ \checkmark\\ \circ\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\circ\\ \circ\\ \checkmark\end{bmatrix}, \quad b\begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix}, b\begin{bmatrix}\checkmark\\ \checkmark\\ \checkmark\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix},$$

$$A\begin{bmatrix}\checkmark\\ \checkmark\\ \checkmark\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\checkmark\\ \checkmark\\ \checkmark\end{bmatrix}, A\begin{bmatrix}\circ\\ \circ\\ \checkmark\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\checkmark\\ \checkmark\\ \checkmark\end{bmatrix}, \quad a\begin{bmatrix}\circ\\ \checkmark\\ \circ\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\circ\\ \checkmark\\ \circ\end{bmatrix}, a\begin{bmatrix}\checkmark\\ \circ\\ \circ\end{bmatrix} \xrightarrow{+} \begin{bmatrix}\circ\\ \checkmark\\ \circ\end{bmatrix}.$$

As for the proof, the observation is *really* obvious: the reason for such behaviour is that when multiplying the considered matrices (one of them 3×3, the other 3×1) terms of lowest degree in one matrix meet terms of lowest degree in the other matrix with no possibility of being cancelled out.

Proposition 2.1. *Let xw (x stands for a single letter) be a non-reducible non–empty word, satisfying the condition:*
in the considered sequence of letters that form w all letters appear in sequences of length ≥ 3 except possibly the whole word w begins with x or xx. Then
1. the possible s–patterns are:

$$\begin{bmatrix} \checkmark \\ \circ \\ \circ \end{bmatrix}, \begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \end{bmatrix} \quad \textit{for } x = b$$

$$\begin{bmatrix} \circ \\ \circ \\ \checkmark \end{bmatrix}, \begin{bmatrix} \circ \\ \checkmark \\ \circ \end{bmatrix} \quad \textit{for } x = B$$

$$\begin{bmatrix} \circ \\ \checkmark \\ \circ \end{bmatrix}, \begin{bmatrix} \checkmark \\ \circ \\ \circ \end{bmatrix} \quad \textit{for } x = a$$

$$\begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \end{bmatrix}, \begin{bmatrix} \circ \\ \circ \\ \checkmark \end{bmatrix} \quad \textit{for } x = A$$

2. In passing from w to xw we decrease the lowest degree exponent if the s–pattern of w is the one given in the first column (the typical one) and we keep it unchanged if the s–pattern is the one given in the second column (the exceptional one).

3. The exceptional cases can only appear for xw when the leftmost letter of w is not x. If this is satisfied and the leftmost letter of w is y, then the s–pattern of sw is exceptional if and only if xy is equal as a formal word to ba, aB, BA or Ab.

Proposition ⇒ Theorem. In effect, the Proposition says that with multiplication from the left by the next matrix in the word, the minimum degree decreases, except in some specifically described cases. These cases are when the letter x begins a new group of letters in the word and the two matrices in the neighbour groups are ba or aB or BA or Ab. This is more than we need to prove the main Theorem — we know that there would be terms of negative exponent in every product that ends with bbb (the need to conjugate to have bbb at the end comes from the fact that with B or A we would have no term of negative exponent to start. On the other hand, a is as good as b). As an example let us consider the word `BBBAAAAbbbaaabbbaaabbbAAABBBAAAbbb`. We show below how the lowest degree in the first column changes when we multiply matrices, starting from the right. The plus sign indicates that the degree goes down after multiplication by the matrix shown above. The plus sign under the first letter (b) indicates that the exponent of the lowest degree term is -1. The space means that there is no change.

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BBBAAAAbbbaaabbbaaabbbAAABBBAAAbbb
++ +++ ++ +++++ +++++++++++ ++ +++
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As may be seen from the above pattern the lowest degree appearing in the first column of the product is -28 which is equal to the difference of the number of group changes of types specified in the Propositions (6) and the number of letters (34) in the word.

Proof of Proposition. Let us consider a word of the form $BAAAw$. We assume by induction that the s–pattern of Aw is $\begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \end{bmatrix}$or $\begin{bmatrix} \circ \\ \circ \\ \checkmark \end{bmatrix}$ for $x = A$. Calculation shows that in both cases the s–pattern of AAw is $\begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \end{bmatrix}$. Moreover, the coefficients of the lowest degree (of exponent, say, d) terms in the first column are equal. The same applies to $AAAw$, additionally we have that also the coefficients of the second lowest degree are equal. Therefore, the first column of $AAAw$ looks as below:

$$\begin{bmatrix} at^d + bt^{d+1} + R_1 \\ at^d + bt^{d+1} + R_2 \\ at^d + bt^{d+1} + R_3 \end{bmatrix},$$

where polynomials R_1, R_2, R_3 contain terms of degree $\geq d+2$, or by convention to be used later:

$$\begin{bmatrix} at^d + bt^{d+1} + \dots \\ at^d + bt^{d+1} + \dots \\ at^d + bt^{d+1} + \dots \end{bmatrix}.$$

Now, we need to consider the s–pattern of $BAAAw$. For that we need to multiply matrices (3×3 by 3×1).

$$\begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix} \cdot \begin{bmatrix} at^d + bt^{d+1} + \dots \\ at^d + bt^{d+1} + \dots \\ at^d + bt^{d+1} + \dots \end{bmatrix}$$

It is just an exercise in matrix multiplication to see that the single lowest degree term will appear in the middle row and will be of degree d as expected. The case of $bAAAw$ is even more obvious.

We describe briefly one more case, that of $Abbbw$. By inductive assumption the s–pattern for bw is $\begin{bmatrix} \checkmark \\ \circ \\ \circ \end{bmatrix}$or $\begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \end{bmatrix}$. In both cases multiplication by b gives pattern $\begin{bmatrix} \checkmark \\ \circ \\ \circ \end{bmatrix}$. One more multiplication by b and we have $\begin{bmatrix} \checkmark \\ \circ \\ \circ \end{bmatrix}$pattern with the exponent of smallest degree term (say, d) smaller by at least 2 than terms in othe rows. This means that the first row of $Abbbw$ looks like

$$\begin{bmatrix} at^d + \dots \\ bt^{d+2} + \dots \\ ct^{d+2} + \dots \end{bmatrix}$$

Now we multiply by A

$$\begin{bmatrix} 0 & 0 & -t^{-1} \\ 0 & -t & -t^{-1} + t \\ -1 & 0 & -t^{-1} + 1 \end{bmatrix} \begin{bmatrix} at^d + \dots \\ bt^{d+2} + \dots \\ ct^{d+2} + \dots \end{bmatrix}$$

Calculation shows that the s–pattern of the product is $\begin{bmatrix} \circ \\ \circ \\ \checkmark \end{bmatrix}$ and the lowest degree exponent is again d as predicted in the Proposition. All other cases are decided in a similar manner which completes the proof of the Proposition and the proof of the Theorem.

Let us observe that in the argument above (for $Abbbw$ and $BAAAw$) we needed the assumption that b or A appears three times to ensure that before we finally multiply by the last matrix (A or B) we have a suitable form of the 3×1 matrix on the right. The first step was to ensure that bw (or Aw) is of a suitable form. It would be sufficient to have just $Abbw$ and $BAAw$ if this was true for some other reason. We will use this observation to obtain a much more general result.
The problem of faithfulness of Burau representation is really reduced to the problem whether A and B generate the trivial group of rank 2. Therefore we need to consider whether the product of matrices arising from a non–reducible formal word in A, a, B, b can be equal to the unit matrix. While the formulation of Theorem 1.1 is certainly more elegant, we can easily obtain the following generalisation.

Theorem 2.2. *Let w be a non-reducible word in letters a, A, b, B such wherever the word contains BA, Ab, ba or aB it is as a part of a bigger sequence of at least B^3A^2, A^3b^2, b^3a^2 or a^3b^2. Then the product of the considered matrices is not the unit matrix.*

We will not go into the details of the proof. There is really no difference from the proof of Theorem 2.1. We need a proposition similar to Proposition 2.1 and we obtain the same simple way to calculate the exponent of the lowest degree term. Once again it may be prudent to assume that the rightmost letter is a b to get the negative degree right at the start.

As an example let us show again a pattern like the one considered earlier, this time for the word `aaaBBabAAAbbbaabAABabbAABabABabbABabbb`. The corresponding pattern is

aaaBBabAAAbbbaabAABabbAABabABabbABabbb
++ ++++++ ++ ++++++++++++++++++++++++

Observe that we deliberately used a long suffix of the form aabAABabbAABabABabbABabbb just to illustrate that it is quite harmless, although it is very far from belonging to $\langle A^3, B^3\rangle$ or even to $\langle A^2, B^2\rangle$ and in fact it has just single letters at twelve positions. In the whole word we have an $aaaBB$ and a $bbbaa$. Those would be forbidden if shorter by just one letter.

Finally, we do not wish to leave the reader under illusion that the sequence of exponents of lowest degree terms might be non increasing in general, so we give an example to the contrary:

bbbbAbbbbbbbbb
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